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LETTER TO THE EDITOR

BRST cohomology for $U_q(sl(2))$ representations

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Abstract. A nilpotent BRST operator Q_q is constructed for the q-deformation of the universal enveloping algebra of sl(2). An associated operator R_q such that $\{Q_q, R_q\} = C_q$ is introduced, where C_q is the q-deformed Casimir invariant $C_q = fe + [\frac{1}{2}h]_q[\frac{1}{2}h+1]_q$. The related operator $Q_q^\dagger$ conjugate to Q_q is also considered. The resulting BRST cohomology of $U_q(sl(2))$ representations is discussed.

The study of 'quantum deformations' of algebraic structures in the theoretical description of two-dimensional systems has received recent impetus from its deep applications in statistical mechanics, conformal field theory and anyonic systems [1]. The q-deformations can alternatively be considered in the case of Lie algebras as generalized dynamical symmetries in quantum mechanical problems [2], in the case of the q-oscillators in terms of generalized Bose or Fermi gases in a thermodynamic context [3], or more fundamentally in terms of non-commutative geometry and quantum planes [4]. The representation theory of the q-deformations is of interest in its own right and adds to the understanding of that of the undeformed algebras.

A natural related question is whether the q-deformations can be used as local gauge symmetries [5] in the spirit of, say, Yang–Mills and Shaw, with the overall aim of physical applications in quantum field theories. In this connection a point of departure is the generic study of gauge theories and their quantization as systems of constraints. Thus, for example, in pure Yang–Mills theory the quantization must implement Gauss' law on physical states [6]; in more general situations the procedure of Dirac [7] is supplanted in modern treatments by the powerful Balatin–Fradkin–Vilkovisky (BFV) formulation of BRST symmetry [8].

In the present letter the problem of gauging the quantum groups is approached from this perspective. In particular we consider the algebraic problem of constructing a BRST symmetry for the simplest quantum universal enveloping algebra, $U_q(sl(2))$. This serves as a zero-dimensional analogue of the associated question of quantization of a local gauged version, and is in itself interesting for the elucidation of the structure of the algebra and its representations. As will be seen, the resulting BRST cohomology is richer than the undeformed case even for $U_q(sl(2))$, and for arbitrary semisimple Lie algebras [9] the deformed $U_q(\mathcal{G})$ cohomology is expected to be similarly richer [10]. The BRST framework is related to studies [11] on the mathematical problem of the Hochschild and cyclic cohomology of quantum groups, but is more natural from the point of view of quantization.

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The foregoing physical considerations allow questions of the definition and uniqueness of the required BRST operator Q_q to be answered for the q-deformed case. Specifically we require a *nilpotent* operator $Q_q^2=0$ for $q\neq 1$ rather than one whose square vanishes in the $q\to 1$ limit, since we are interested in the availability of the BFV-BRST quantization for constraints arising as a q-deformed algebra. Secondly, we demand that Q_q be a 'minimal' extension of the standard undeformed BRST operator Q for sl(2) (see below), whose $q\to 1$ limit is just Q itself.

For purposes of comparison we quote the form of the well known (nilpotent) BRST operator Q in the case of a compact semisimple or reductive Lie algebra. In a Hermitian basis with generators T_a and commutation relations

$$[T_a, T_b]=\mathrm{i}C^c_{ab}T_c \tag{1}$$

to which are appended the ghost and antighost generators c^a, $\bar{c}_b$ respectively, which satisfy the Clifford algebra

$$\{c^a, \bar{c}_b\}=\delta^a_b \tag{2}$$

the corresponding BRST operator is defined by

$$Q=c^aT_a+\tfrac{1}{2}\mathrm{i}c^cc^bC^a_{bc}\bar{c}_a. \tag{3}$$

For sl(2) it will be more convenient to use the standard (Cartan–Weyl) generators e, f and h. In the deformed case they satisfy

$$\begin{aligned}[][e,f]&=[h]_q\\ [h,e]&=2e\\ [h,f]&=-2f\end{aligned} \tag{4}$$

where

$$[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}}.$$

To e, f and h are associated respectively the ghost generators c^+, c^- and c^0 respectively, which satisfy the anticommutation relations (2) with their antighost counterparts. With these preliminaries the required *deformed* operator Q_q for $\mathrm{U}_q(\mathrm{sl}(2))$ is given by

$$\begin{aligned}Q_q=ec^-+fc^++[h]c^0+\bar{c}^0c^+c^-+(q^{h+1}+q^{-h-1})\bar{c}^-c^-c^0-(q^{h-1}+q^{-h+1})\bar{c}^+c^+c^0\\ +(q-q^{-1})^2[h]\bar{c}^+\bar{c}^-c^+c^-c^0.\end{aligned} \tag{5}$$

Using (2) and (4) the nilpotency of Q_q can be verified directly,

$$Q_q^2=0$$

which constitutes our central result. The form of Q_q is fixed from the nilpotency requirement once it is assumed to be linear in e and f in $\mathrm{U}_q(\mathrm{sl}(2))$ (and, of course, to have unit ghost number). It then is the unique minimal extension of the undeformed Q, which reads explicitly

$$Q=ec^-+fc^++hc^0+\bar{c}^0c^+c^-+2\bar{c}^-c^-c^0-2\bar{c}^+c^+c^0 \tag{6}$$

which is polynomial in $q^h\equiv k$ and has the correct $q\to 1$ limit $Q_q\to Q$. Note also that the ghost generators have been defined with *standard* fermionic anticommutation relations rather than in terms of q-deformed operators, in conformity with the strategy that in the field theory case the BFV-BRST quantization should reproduce the correct *ghost fields to represent the required Faddeev-Popov determinants arising from gauge*-fixing. (This requirement also ensures the simplifying feature that any operator of fixed ghost number can be expanded in a finite number of terms monomial in the ghost and antighost generators, with coefficients in the q-deformed universal enveloping algebra.)

The operator Q_q thus defined must be accompanied by some additional structure if its utility in discussions of cohomology is to be comparable with the corresponding undeformed operator (3) or, say, the exterior derivative in the differential geometry case. The analogue of the Laplacian of the latter case is, for the algebraic case, the Casimir invariant or a generalization thereof [9]. Here again the q-deformed situation is fraught with some ambiguity in that, even for the lowest order, different constructions lead to formally distinct expressions which differ in the $q \to 1$ limit from the standard Casimirs by (sometimes divergent) overall constants [12]. In the case of $U_q(\mathrm{sl}(2))$ two commonly discussed alternatives [13] are

$$C_q = fe + \left[\frac{h+1}{2}\right]_q^2 \tag{7}$$

and

$$C_q = fe + \left[\frac{h}{2}\right]_q \left[\frac{h+2}{2}\right]_q \tag{8}$$

respectively. In the following, we mainly consider the second choice, as its $q \to 1$ limit is the standard $sl(2)$ Casimir with eigenvalue $j(j+1)$, and it is this operator which is involved in the corresponding construction in the undeformed case [9], whereas the first contains the additive constant $\frac{1}{4}$ in the limit. The analogy with the differential case is completed by introducing an operator R_q whose anticommutator with Q_q is the desired Casimir operator, $\{Q_q, R_q\} = C_q$. Again, since R_q has ghost number -1, it has a finite expansion, and the form of Q_q dictates the unique solution

$$\begin{aligned} R_q = f\bar{c}^- + e\bar{c}^+ + [h]\Bigg\{ & \frac{(q+q^{-1})}{(q-q^{-1})}\,\bar{c}^0 + \left(\frac{q^{(h/2)-1} - q^{-(h/2)+1}}{q^{(h/2)-1} + q^{-(h/2)+1}}\right)\bar{c}^0\bar{c}^+c^+ \\ & - \left(\frac{q^{(h/2)+1} - q^{-(h/2)-1}}{q^{(h/2)+1} + q^{-(h/2)-1}}\right)\bar{c}^0\bar{c}^-c^- \\ & - \frac{2(q^2-q^{-2})}{(q^{(h/2)-1}+q^{-(h/2)+1})(q^{(h/2)+1}+q^{-(h/2)-1})}\,\bar{c}^+\bar{c}^-\bar{c}^0c^+c^- \Bigg\} \end{aligned} \tag{9}$$

where the Casimir (8) has been used.

By standard arguments [9] it follows from the above structure that the only non-trivial cocycles in the cohomology of Q_q are the representations of $U_q(\mathrm{sl}(2))$ with vanishing Casimir invariant. At generic values of q, and ghost number zero, these are just the singlet representations (with spin zero), as in the undeformed case. At non-zero ghost number the analysis also follows the undeformed case [9], so that there are for example the corresponding singlets at maximum ghost number (=3 for sl(2)). More general possibilities are apparent when the structure of the constraint condition $Q_q|phys\rangle = 0$ is examined at fixed ghost number (say, 0). In particular the requirement $[h]_q c^0|phys\rangle = 0$ suggests that, for non-generic values of q, and for h having a suitable spectrum, non-singlet solutions may occur.

For q a root of unity the representations of $U_q(\mathrm{sl}(2))$ can be described as follows [14]. Let M be the smallest positive integer such that $q^M=1$, and $N=M$ (M odd) or $M/2$ (M even). Define $X=(\delta_{ij+1(\mathrm{mod}\,N)})$, $Z=(q^i\delta_{ij})$, $i,j\in Z_N$, so that $ZX=qXZ$. Now up to a trivial rescaling $e\to x^{-1}e, f\to xf$ the matrices

$$e=[aZ]X \qquad f=[bZ^{-1}]X^{-1} \qquad k=q^h=abZ^2$$

provide an N-dimensional representation of $U_q(\mathrm{sl}(2))$, where a, b are numbers, and $[u]=(u-u^{-1})/(q-q^{-1})$. When neither a nor b is an integer power of q, the representation is irreducible and periodic ($\det(e)\neq 0, \det(f)\neq 0$), and irreducible and semiperiodic when only one of them is (so that only one of the determinants vanishes). When both a and b are an integer power of q, there is the possibility that the representation is indecomposable.

For the Casimir given by (8) the eigenvalue in such an N-dimensional representation is

$$\begin{aligned} C_q &= (abq+(abq)^{-1}-q-q^{-1})/(q-q^{-1})^2 \\ &= q(ab)^{-1}(ab-1)(ab-q^{-2})/(q-q^{-1})^2. \end{aligned} \tag{10}$$

Therefore it constitutes a cohomology class if $ab=1$ or q^{-2}. This certainly includes the periodic case, but excludes the semiperiodic case in which only *one* of a and b must be a power of q. Examples of the final category (*both* a and b powers of q such that $ab=1$ or q^{-2}), are the so-called indecomposable spin representations with spin $j=\frac{1}{2}N-1$ (for odd $N\geqslant 3$) [14]. It is clear that the structure is richer than that of the undeformed case even for this simplest example of $U_q(\mathrm{sl}(2))$ and at zero ghost number; this feature would be expected more generally and especially in the case of $U_q(\mathscr{G})$ [10].

As pointed out by van Holten in the case of compact Lie algebras [9], it is quite natural to introduce a BRST complex with the structure of supersymmetric quantum mechanics in which the BRST operator Q is augmented by its adjoint $Q^\dagger$; their anticommutator plays the role of the Hamiltonian H and is a certain 'BRST completion' of the Casimir invariant. In this case there is, for unitary representations, by the positivity of H, a Hodge decomposition, and a complete decomposition of states into supermultiplets at each ghost number is possible. In a Hermitian basis for the Lie algebra, the adjoint of Q (which is also nilpotent) is given by

$$Q^\dagger=\bar{c}_aT^a+\tfrac{1}{2}\mathrm{i}\bar{c}_b\bar{c}_cC_a^{cb}c^a \tag{11}$$

where T^a and T_a are dual WRT to the Killing form. We have so far been unable to obtain a q-analogue of this object. Nevertheless, one can define a $\dagger$ operation for $U_q(\mathrm{sl}(2))$ via the substitutions

$$\begin{aligned} &e\to f \qquad f\to e \qquad h\to h \\ &c^0\leftrightarrow\bar{c}^0 \qquad c^+\leftrightarrow\bar{c}^+ \qquad c^-\leftrightarrow\bar{c}^- \end{aligned}$$

in Q_q to obtain a nilpotent $Q_q^\dagger$

$$\begin{aligned} Q_q^\dagger &= f\bar{c}^-+e\bar{c}^++[h]\bar{c}^0+\bar{c}^-\bar{c}^+c^0+(q^{h+1}+q^{-h-1})\bar{c}^0\bar{c}^-c^--(q^{h-1}+q^{-h+1})\bar{c}^0\bar{c}^+c^+ \\ &\quad+(q-q^{-1})^2[h]\bar{c}^0\bar{c}^-\bar{c}^+c^-c^+. \end{aligned} \tag{12}$$

From Q_q and $Q_q^\dagger$ we can define the object $H=\{Q_q, Q_q^\dagger\}$ which commutes with both Q_q and $Q_q^\dagger$ and can be utilized in the same way as van Holten's in the analysis of cohomology. Note however that

$$\{Q_q, Q_q^\dagger\}=\left(\frac{q^h-q^{-h}}{q-q^{-1}}\right)^2+\left(\frac{q^h-q^{-h}}{q-q^{-1}}\right)+2fe+\dots \tag{13}$$

where ... stands for purely ghost-dependent terms, and thus *H cannot* be considered as the BRST completion of the Casimir.

In conclusion, it has been shown that within a framework appropriately motivated by consideration of quantization of systems with first class constraints generating a q-deformed algebra, a nilpotent BRST operator Q_q can be defined. Q_q and related operators have been constructed for $U_q(sl(2))$ and their properties verified using symbolic computation; more systematic studies of more general algebras are also possible [10]. Although the analogy with the de Rham cohomology is not straightforward (compare [11]), enough of the structure has been established to show that the BRST cohomology of $U_q(sl(2))$ is richer than the undeformed case; for example there are non-trivial irreducible representations at q a root of unity which correspond to non-trivial cocycles (cohomology classes). This feature is expected to persist for more general algebras [10].

The departure of the q-deformed from the undeformed case, and in particular the differences between $Q_q^\dagger$ and R_q, point perhaps to the need for a more detailed understanding and improved definition of the adjoint operation $\dagger$. This is perhaps not surprising if the Casimir invariant and Killing form are looked at from a geometrical perspective: presumably in the q-deformed case there is a need to apply considerations of 'q-geometry' [4].

The basic conclusion of the present letter is that the BRST cohomology is interesting in the q-deformed case. This has implications for physics based on gauged quantum groups. Presumably the physical states need no longer be 'gauge invariant' (that is, singlet under the constraint algebra at zero ghost number); instead appropriate multiplets of some (global?) algebra might be allowed, suggesting fascinating possibilities for example for the family replication problem in the context of unified models.

After this work was completed, we became aware of the paper of Kunz *et al* [15] which also studies the problem of constructing quantum BRST operators. However, they deal with the 'twisted SU(2) group' of Woronowicz [16] while we deal with $U_q(sl(2))$ as defined by Drinfeld and Jimbo. Furthermore, they employ q-deformed ghosts while we prefer to stick to undeformed ones. The relation between our two approaches remains to be clarified.

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